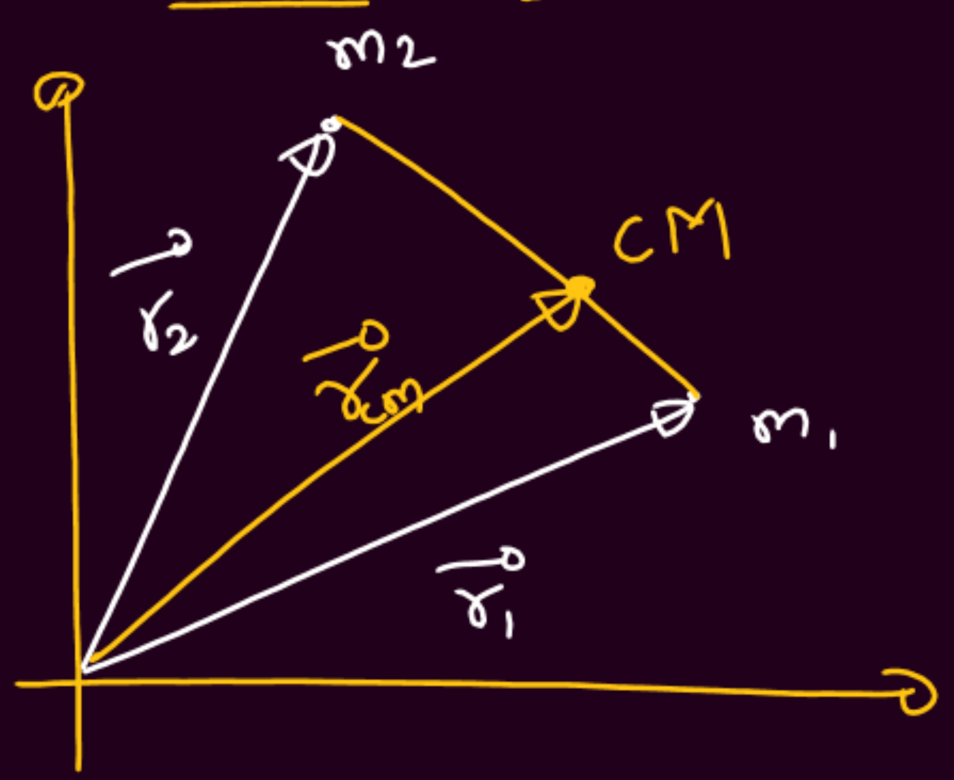
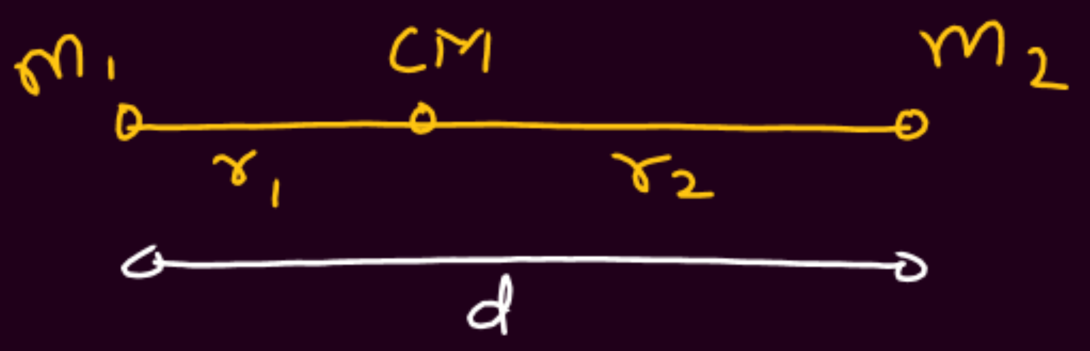
Center of Mass



$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$X_{cm} = \frac{m_1 X_1 + m_2 X_2}{m_1 + m_2}$$



$$\boxed{r_1 = \frac{m_2 d}{m_1 + m_2}}$$

$$r_2 = \frac{m_1 d}{m_1 + m_2}$$

$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

$$\boxed{m_1 r_1 = m r_2}$$

{ if r_1 & r_2 are distances of m_1 & m_2 from CM

→ CM is closer to Heavier

Q. 2 kg —— 8 kg

$x = 2\,cm$ $\quad$ $x = 8\,cm$

$$X_{cm} = \frac{(2)(2) + 8(8)}{2+8}$$

$$= \underline{6.8\,cm}$$

Q. 4 kg —— 6 kg

$x = -2$ $\quad$ $x = 12$

$$X_{cm} = \frac{4(-2) + 6(12)}{4+6}$$

$$= \underline{\underline{6.4}}$$

Q. 4 kg —— CM —— $m = ?$

$x = 4$ $\quad$ $x = 8$ $\quad$ $x = 10$

$$8 = \frac{4(4) + 10m}{4+m}$$

$$32 + 8m = 16 + 10m$$

$$16 = 2m$$

$$\boxed{m = 8\,kg}$$

Q. 2 kg —— 4 kg —— 14 kg

$x = 2m$ $\quad$ $x = 6m$ $\quad$ $x = 10m$

$$X_{cm} = \frac{2(2) + 4(6) + 14(10)}{2+4+14} = \frac{168}{20} = \underline{8.4\,m}$$

Q. 2 kg $(1, 3)$

3 kg $(4, -1)$

$$x_{cm} = \frac{2(1) + 3(4)}{2+3}$$

$$= \frac{14}{5}$$

$$y_{cm} = \frac{2(3) + 3(-1)}{2+3}$$

$$= \frac{3}{5}$$

$$CM\left(\frac{14}{5}, \frac{3}{5}\right)$$

Case

$$\boxed{m_1 x_1 = m_2 x_2}$$

Regular Geometries with uniform mass distribution

CM $\rightarrow$ geometric center

CM of Extended objects

$$X_{cm} = \frac{M_1 x_1 + M_2 x_2}{M_1 + M_2}$$

Both objects have same density

$$X_{cm} = \frac{l_1 x_1 + l_2 x_2}{l_1 + l_2}$$

$$X_{cm} = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2}$$

$$X_{cm} = \frac{V_1 x_1 + V_2 x_2}{V_1 + V_2}$$

① Semicircular Arc

$$\boxed{y = \frac{2R}{\pi}}$$

② Semicircular Disc

$$\boxed{y = \frac{4R}{3\pi}}$$

③ Hollow Hemisphere

$$y = \frac{R}{2}$$

④ Solid Hemisphere

$$y = \frac{3R}{8}$$

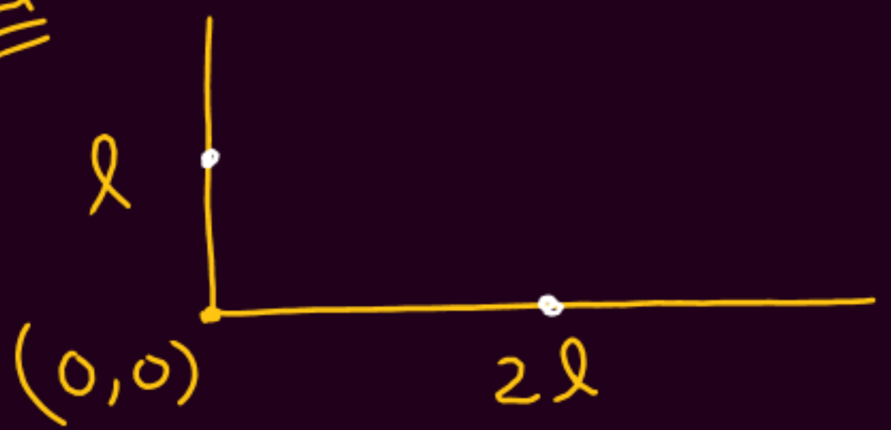
⑤ Hollow Cone

$$y = \frac{h}{3}$$

⑥ Solid Cone

$$y = \frac{h}{4}$$

Q



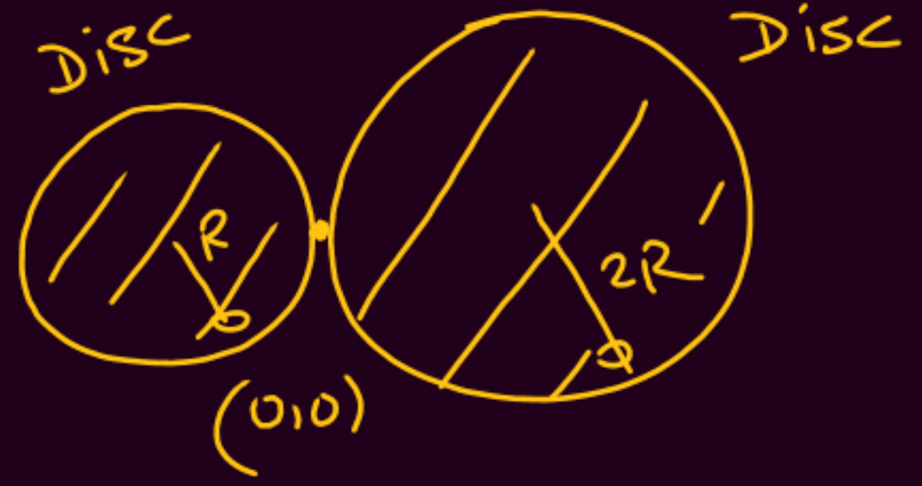
$l \; (0, \frac{l}{2})$

$2l \; (l, 0)$

$$X_{cm} = \frac{l(0) + 2l(l)}{l + 2l} = \frac{2l}{3}$$

$$Y_{cm} = \frac{l(\frac{l}{2}) + 2l(0)}{l + 2l} = \frac{l}{6}$$

Q.



Small $(-R, 0)$

Big $(2R, 0)$

$$X_{cm} = \frac{\pi R^2(-R) + \pi (2R)^2(2R)}{\pi R^2 + \pi (2R)^2} = \frac{7R}{5}$$

$\boxed{Y_{cm} = 0}$

Q.

Small $(-R, 0)$

Big $(2R, 0)$

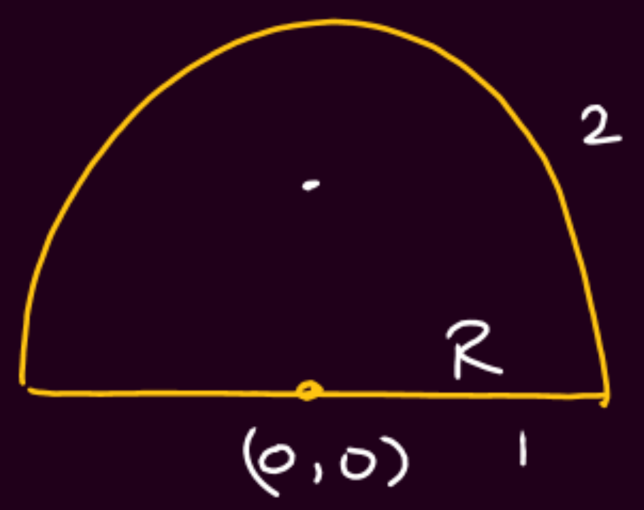
$Y_{cm} = 0$

$$X_{cm} = \frac{(V)(-R) + (8V)(2R)}{V + 8V}$$

$$= \frac{15}{9} R$$

$$X_{cm} = \frac{5R}{3}$$

Semicircular frame



$1\ (0, 0)$

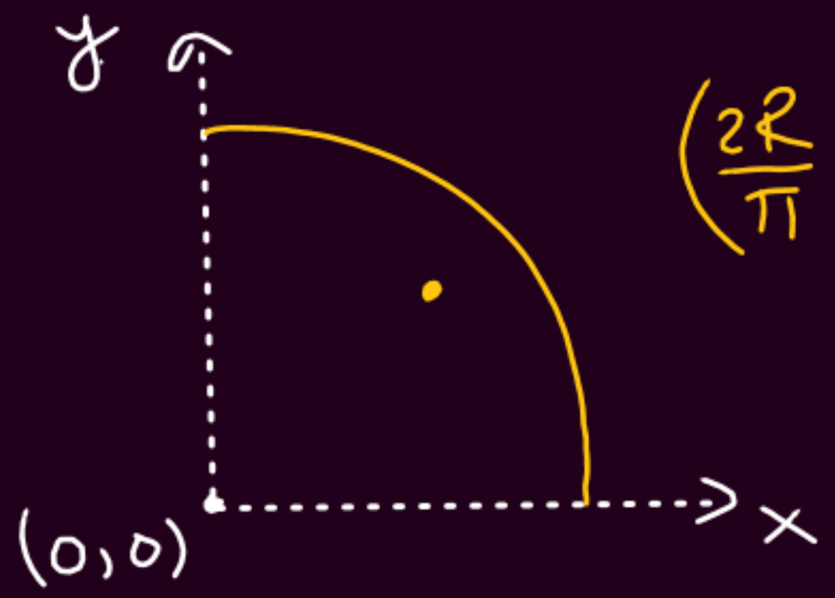
$2\ \left(0, \frac{2R}{\pi}\right)$

$$X_{cm} = 0$$

$$Y_{cm} = \frac{2R(0) + \pi R\left(\frac{2R}{\pi}\right)}{2R + \pi R}$$

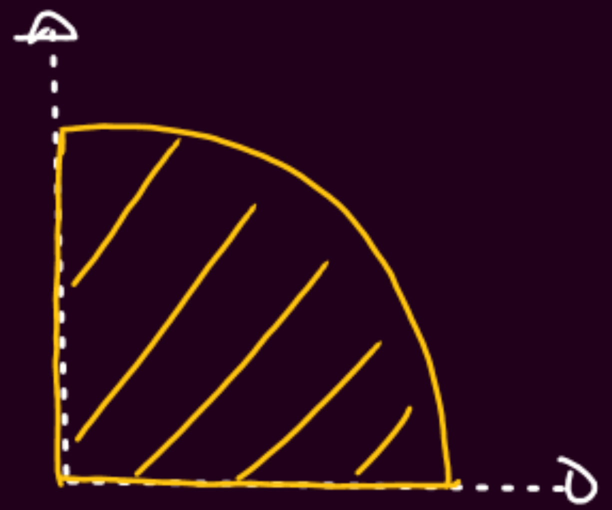
$$Y_{cm} = \frac{2R}{2+\pi}$$

Quarter Circular Arc



$$\left(\frac{2R}{\pi}, \frac{2R}{\pi}\right)$$

Quarter Circular Disc



$$\left(\frac{4R}{3\pi}, \frac{4R}{3\pi}\right)$$

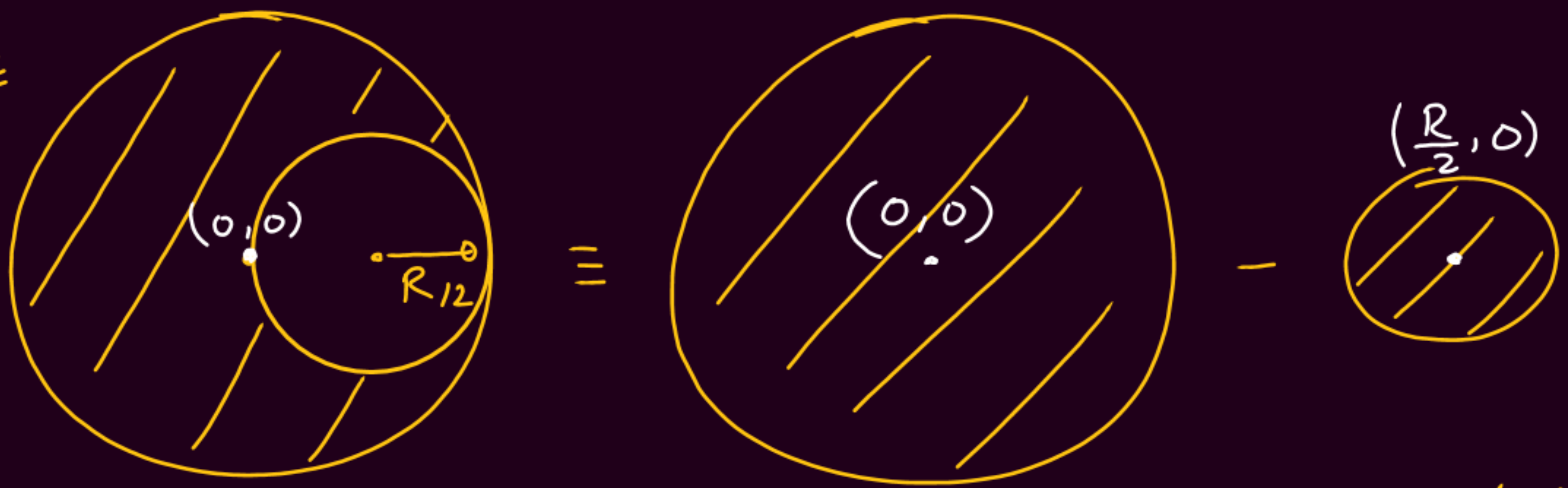
If Some Part is Removed

Initially M (x_1)

removed m (x_2)

$$X_{cm} = \frac{M x_1 - m x_2}{M - m}$$

Q.



$A_1 = \pi R^2$

$A_2 = \pi\left(\frac{R}{2}\right)^2 = A_1/4$

$y_{cm} = 0$

$$X_{cm} = \frac{A_1(0) - A_2(R/2)}{A_1 - A_2}$$

$$X_{cm} = \frac{-A_1/4\,(R/2)}{A_1 - A_1/4}$$

$$X_{cm} = -\frac{R}{6}$$

Motion of CM

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots}{m_1 + m_2 + \cdots}$$

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots}{m_1 + m_2}$$

$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots}{m_1 + m_2 + \cdots}$$

$$\vec{p}_{sys} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots$$

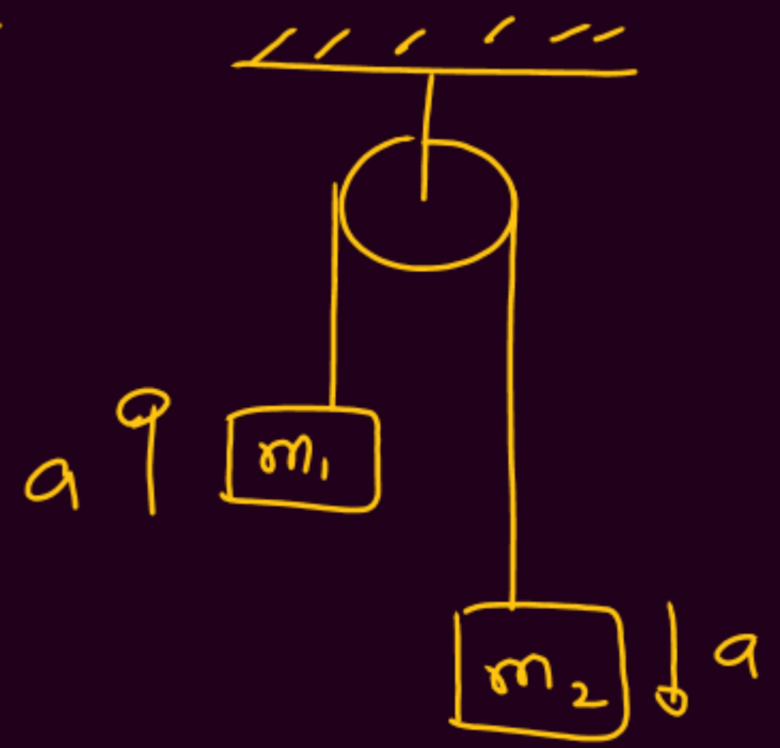
Q. 1 kg $\quad \vec{v}_1 = (\hat{i} + 2\hat{j})$

2 kg $\quad \vec{v}_2 = (-2\hat{i} + 3\hat{j})$

$$\vec{v}_{cm} = \frac{1(\hat{i} + 2\hat{j}) + 2(-2\hat{i} + 3\hat{j})}{1+2}$$

$$\vec{v}_{cm} = \frac{-3\hat{i} + 8\hat{j}}{3}$$

Case



① Acc. of System / Blocks

$$\boxed{a = \frac{(m_2 - m_1) g}{m_1 + m_2}}$$

② Acc. of CM

$$\boxed{a_{cm} = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)^2 g}$$

$$a_{cm} = \frac{m_2 a + m_1(-a)}{m_1 + m_2}$$

$$a_{cm} = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) a$$

$$a_{cm} = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) \frac{m_2 - m_1}{m_1 + m_2} g$$

Law of Conservation of Linear Momentum

If $\vec{F}_{ext} = 0$

$\vec{p}_{sys} = \text{Const.}$

$$(\vec{p}_1 + \vec{p}_2 + \cdots)_f = (\vec{p}_1 + \vec{p}_2 + \cdots)_i$$

Recoil of Gun

m_2 m_1

$v_2 \leftarrow \square \quad \square \rightarrow v_1$

① $m_1 v_1 = m_2 v_2$

Recoil speed (Gun)

② Recoil Energy (Gun) $= \frac{1}{2} m_2 v_2^2$

③ $\frac{K_2}{K_1} = \frac{m_1}{m_2}$

$K = \frac{p^2}{2m}$

Bomb at Rest
Explodes into 2 parts

α- emisson

A

$v_2 \leftarrow (A-4) \quad \alpha \rightarrow v_\alpha$

① $(A-4) v_2 = 4 v_\alpha$

② $\frac{K_{A-4}}{K_\alpha} = \frac{4}{A-4}$

③ $K_\alpha = \frac{A-4}{A} Q$

Q. Gun → 2 kg
Bullet → 50 g, 400 m/s

① Recoil speed

$$2v = \frac{50}{1000} \times 400$$

$$v = 10\ m/s$$

② Recoil Energy

$$= \frac{1}{2}(2)(10)^2 = 100\ J$$

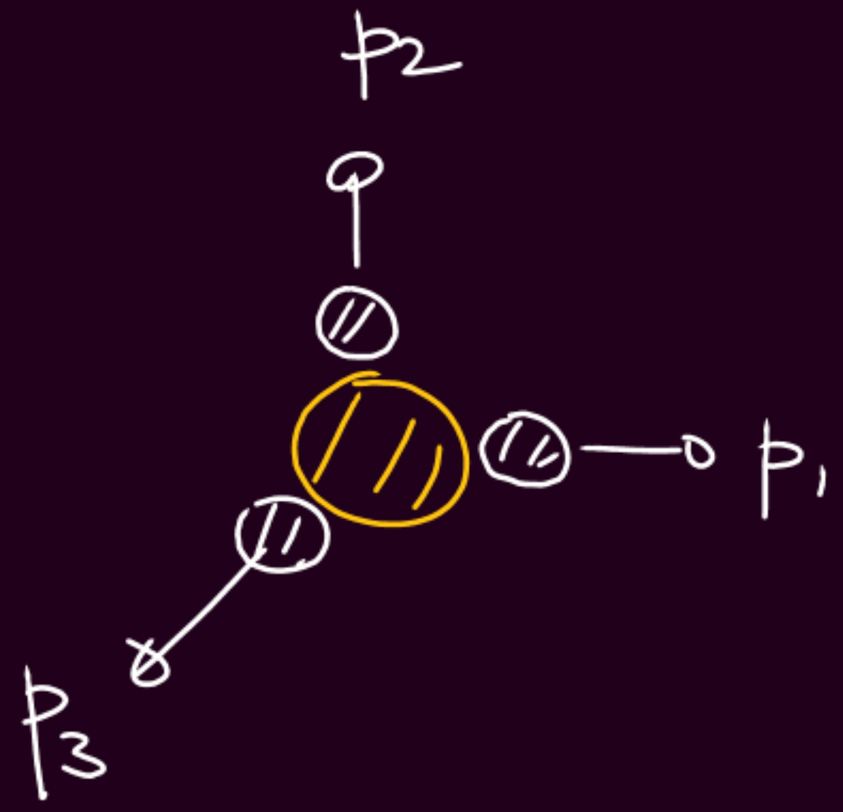
Q. Bomb at Rest → Explodes in 2 (2:3)
lighter K = 60 J

$$\frac{K_2}{60} = \frac{K_2}{K_1} = \frac{2}{3}$$

$$K_2 = 40\ J$$

Bomb at Rest

- Explodes into 3
- 2 goes $\perp^r$



$$\boxed{p_3^2 = p_1^2 + p_2^2}$$

Energy released in explosion

$= K_1 + K_2 + K_3$

Q: masses $(1:2:3)$

lighter $\to \perp^r$ (10 m/s each)

$$(3mv)^2 = (10m)^2 + (10(2m))^2$$

$$3mv = 10m\sqrt{5}$$

$$\boxed{v = \frac{10}{3}\sqrt{5} \text{ m/s}}$$

Q. A & B

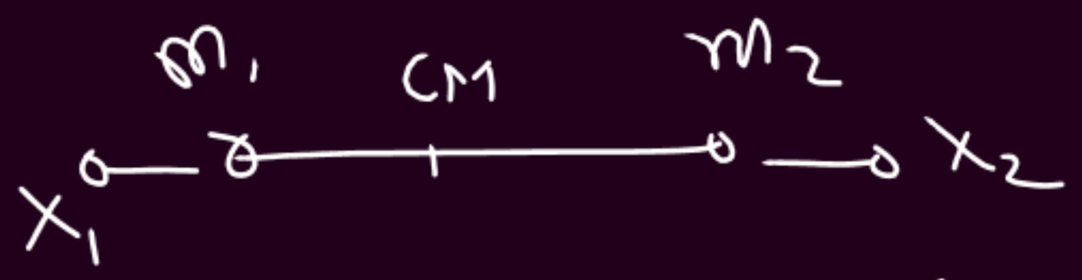
Start moving → mutual Attraction

$v_A = v$

$v_B = 2v$

$v_{cm} = ?$

$v_{cm} = 0$ Rest



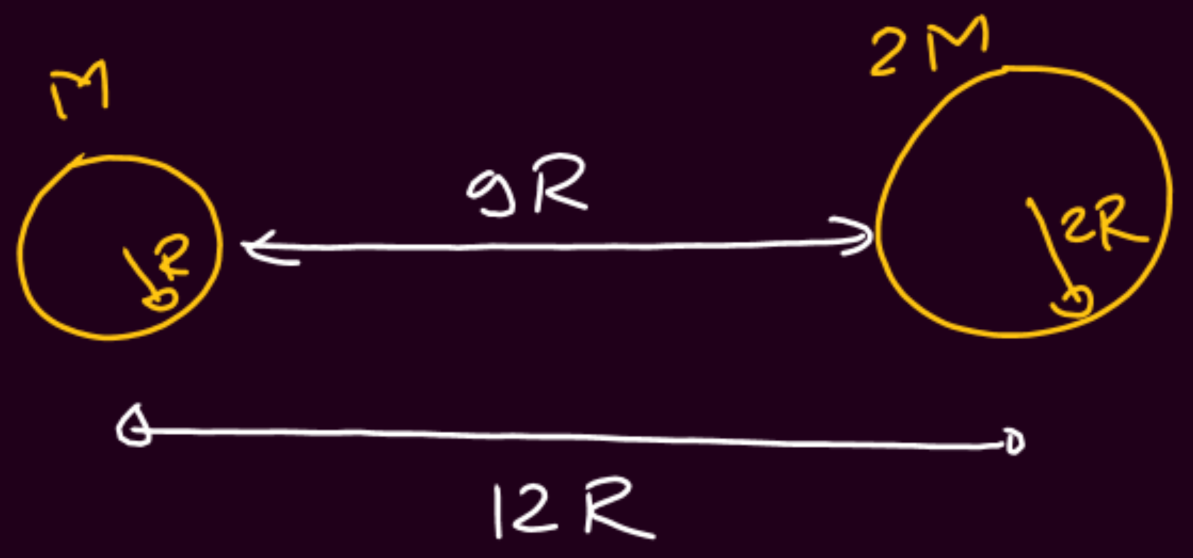
$m_1 x_1 = m_2 x_2$

$x_1 + x_2 = d$

$$\boxed{x_1 = \frac{m_2 d}{m_1 + m_2}}$$

d = Rel. Displacement

Q.



→ Start moving → mutual attraction

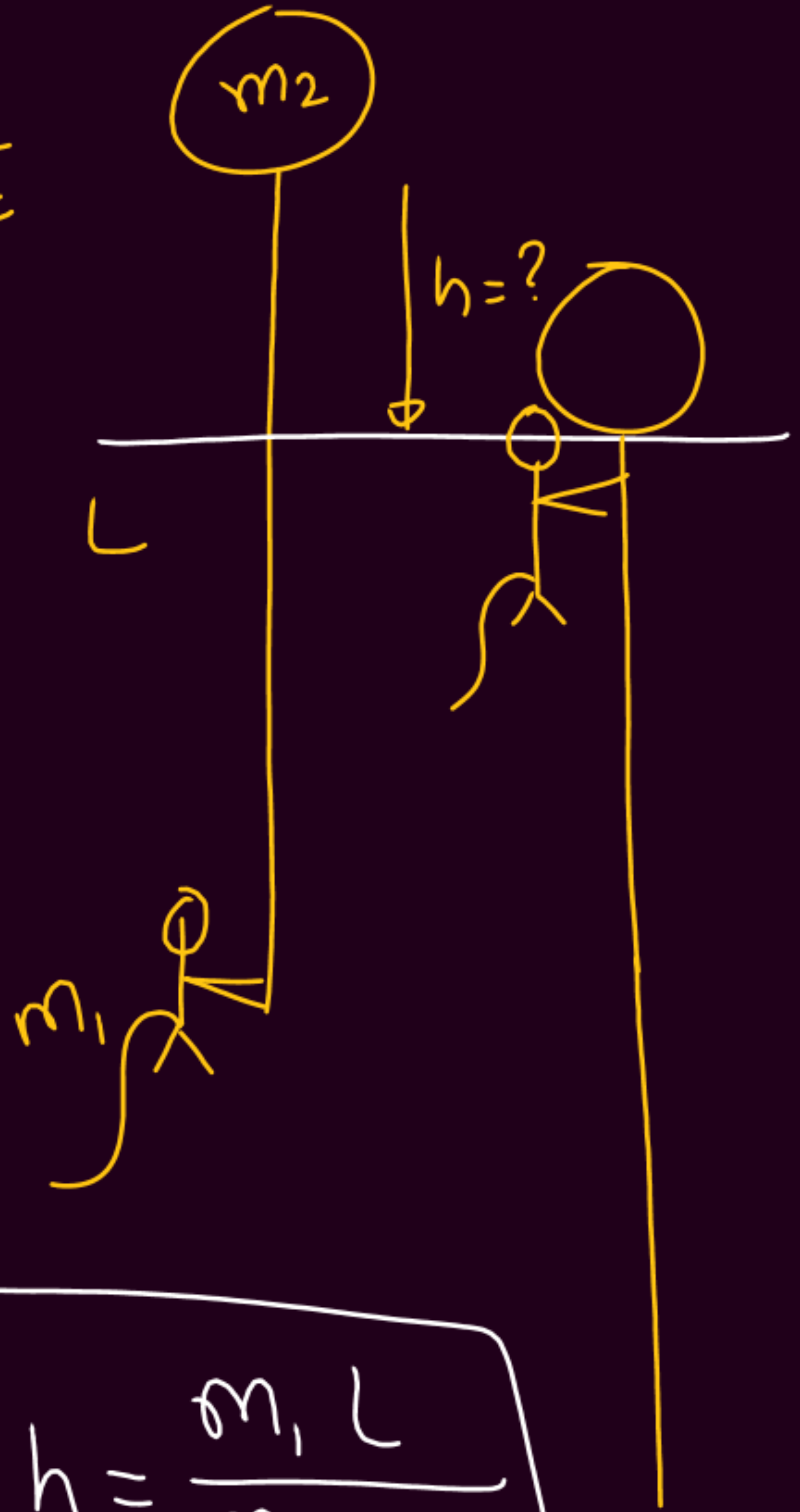
→ Dist by lighter till collision

$d_{rel} = 9R$

$$x_1 = \frac{m_2 d}{m_1 + m_2} = \frac{(2M)(9R)}{3M} = 6R$$

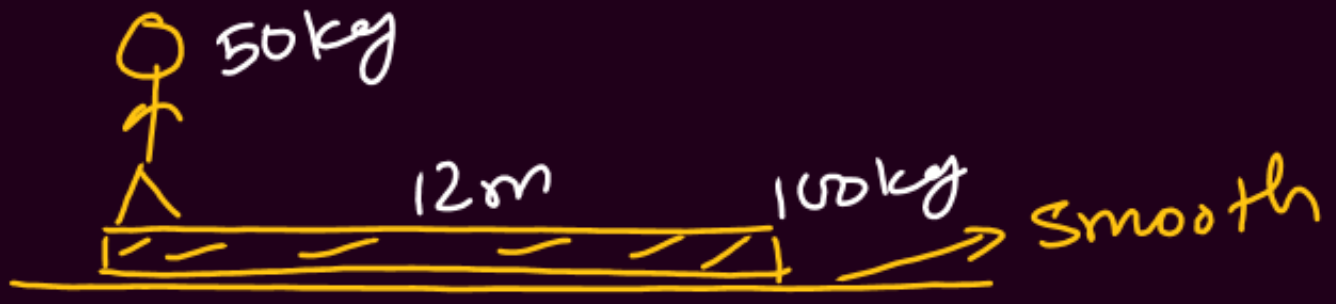
Q.

Rest



$$\boxed{h = \frac{m_1 L}{m_1 + m_2}}$$

Q. Planck



① goes to center

$$x_p = \frac{(50)(6)}{50+100} = 2m$$

② goes to other End

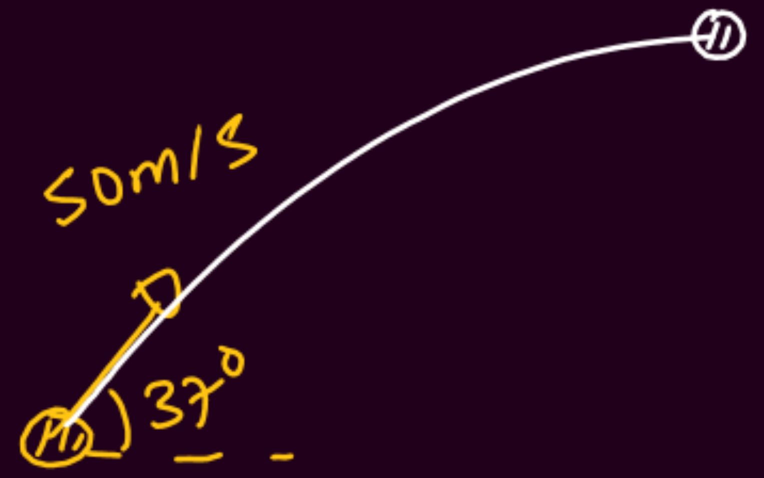
$$x_p = \frac{50(12)}{50+100} = 4m$$

③ Man starts walking with 3m/s

$$\boxed{v_2 = \frac{m_1 v_{rel}}{m_1 + m_2}}$$

$$v_p = \frac{(50)(3)}{50+100} = 1m/s$$

Q.



→ Explodes into 2 (1:2)

→ Lighter → Rest

$$p_{\text{Just Before}} = p_{\text{Just After}}$$

$$3m(50\cos 37^\circ) = m(0) + 2m(v)$$

$$\boxed{v = 60\,m/s}$$

Q.

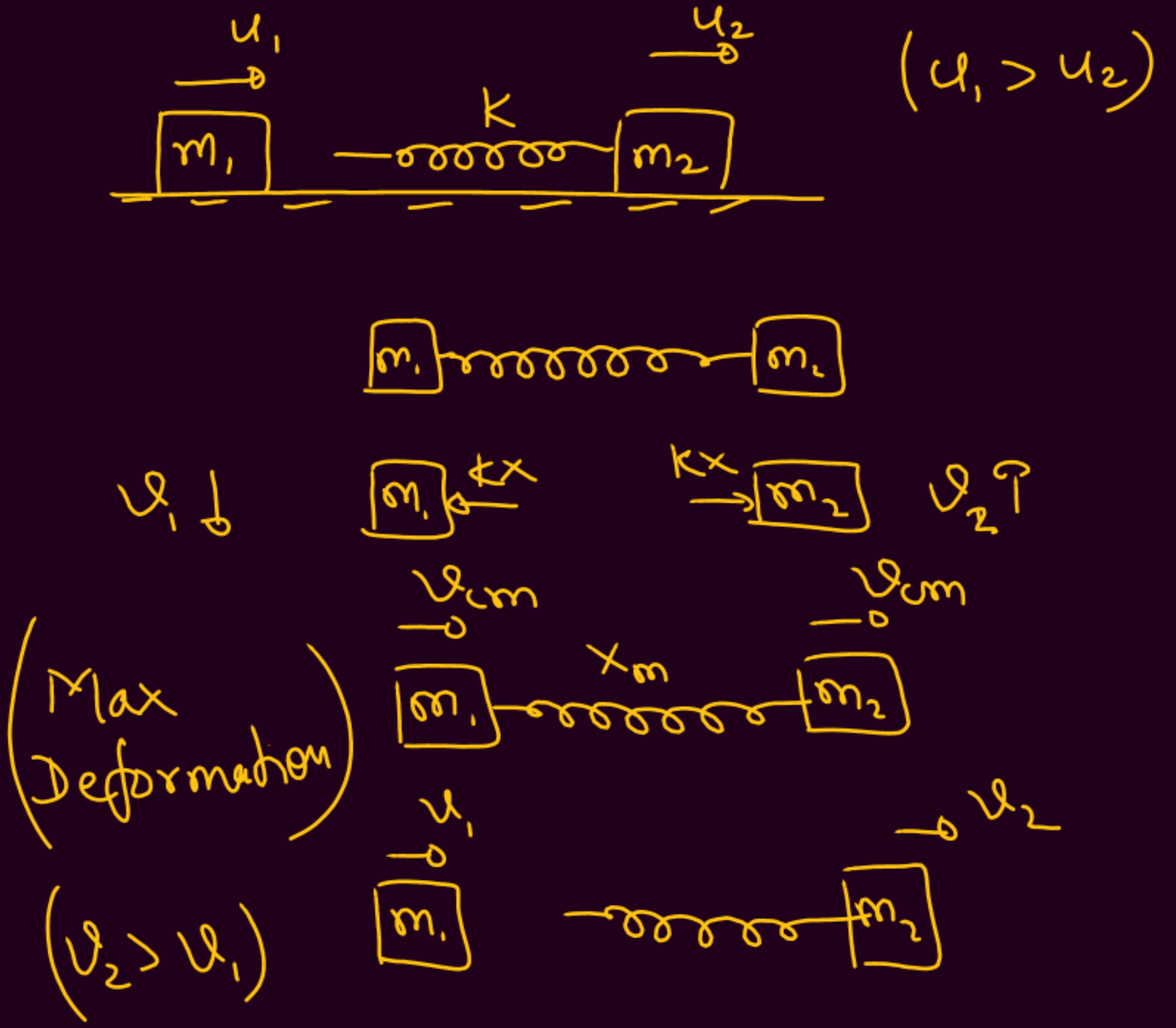
Position of CM in 2s

① $v_{cm} = \dfrac{3(10) + 2(-5)}{3+2} = 4\,m/s$

② $x_{cm_i} = \dfrac{3(2) + 2(22)}{5} = 10\,m$

Ans $= 10 + (4\times 2) = 18\,m$

Extra: Spring-Block System (Collision Theory)



① $\begin{pmatrix} \text{KE in CM} \\ \text{frame} \end{pmatrix} = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(v_1 - v_2)^2$

② In case of max deformation

→ $v_{rel} = 0$

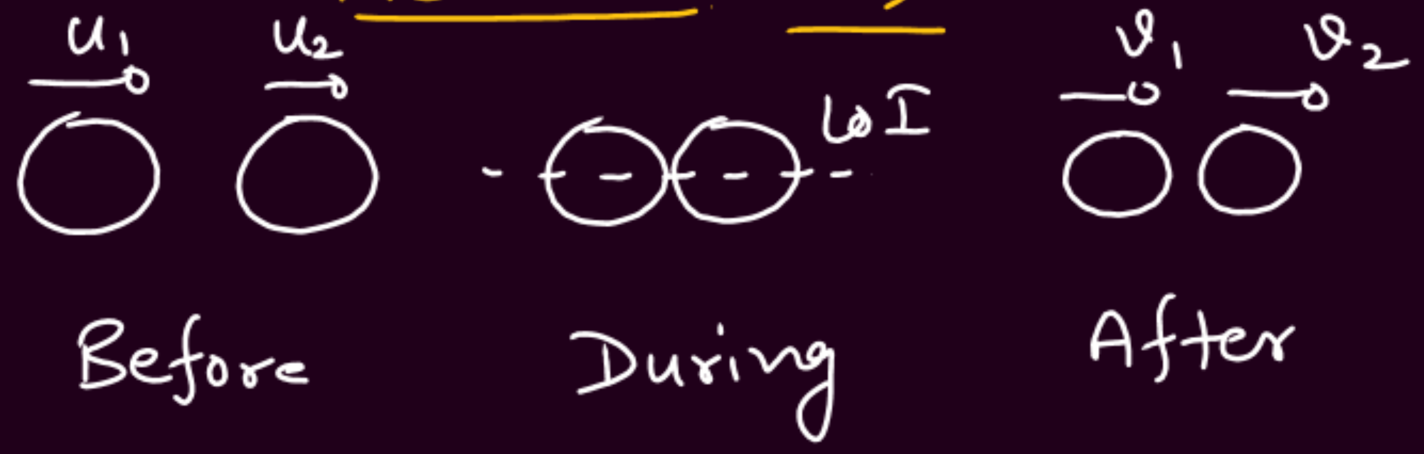
→ $v_1 = v_2 = v_{cm}$

→ $KE_{CM} = 0$

→ $\frac{1}{2}k x_m^2 = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(u_1 - u_2)^2$

Collision

Head-on (1D)



Oblique (2D/3D)

Line of Impact

Coefficient of Restitution

$$e = \frac{\text{Vel. of Sep.}}{\text{Vel. of App.}} = \frac{\text{Energy of Reformation}}{\text{Energy of Deformation}}$$

$$\boxed{e = \frac{v_2 - v_1}{u_1 - u_2}}$$

$$0 \le e \le 1$$

$e = 1$ Elastic

$0 \le e < 1$ Inelastic

$\hookrightarrow$ $\boxed{e = 0}$ Perfectly Inelastic

Elastic $(e=1)$

① $(\vec{p})_{\text{just Before}} = (\vec{p})_{\text{During}} = (\vec{p})_{\text{Just After}}$

② $(ME)_{JB} = (ME)_{\text{During}} = (ME)_{JA}$

③ $KE_{\text{Before}} = KE_{\text{After}}$

④ Loss in $KE = 0$

Inelastic $(0 \leq e < 1)$

① $(\vec{p})_{\text{just Before}} = (\vec{p})_{\text{During}} = (\vec{p})_{\text{just After}}$

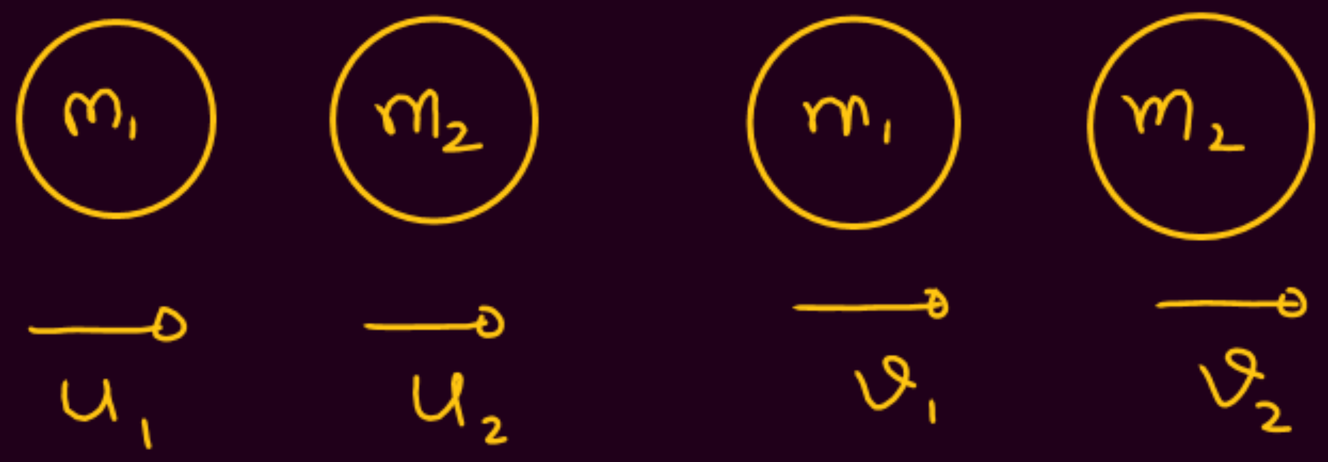
② ME → may or may not be conserved

③ $KE_{\text{Before}} > KE_{\text{After}}$

④ $\begin{pmatrix}\text{Loss in} \\ KE\end{pmatrix} = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(u_1-u_2)^2(1-e^2)$

↳ Deformation (PE) / Sound / Heat etc.

Head-on Elastic



① $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$

② $\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

③ $e = 1$

$u_1 - u_2 = v_2 - v_1$

④ $$\boxed{v_1 = u_1 + \frac{2m_2}{m_1 + m_2}(u_2 - u_1)}$$

$$v_2 = u_2 + \frac{2m_1}{m_1 + m_2}(u_1 - u_2)$$

Special Case if $m_1 = m_2 = m$

$$\left.\begin{aligned} v_1 &= u_2 \\ v_2 &= u_1 \end{aligned}\right\} \text{velocity Exchange}$$

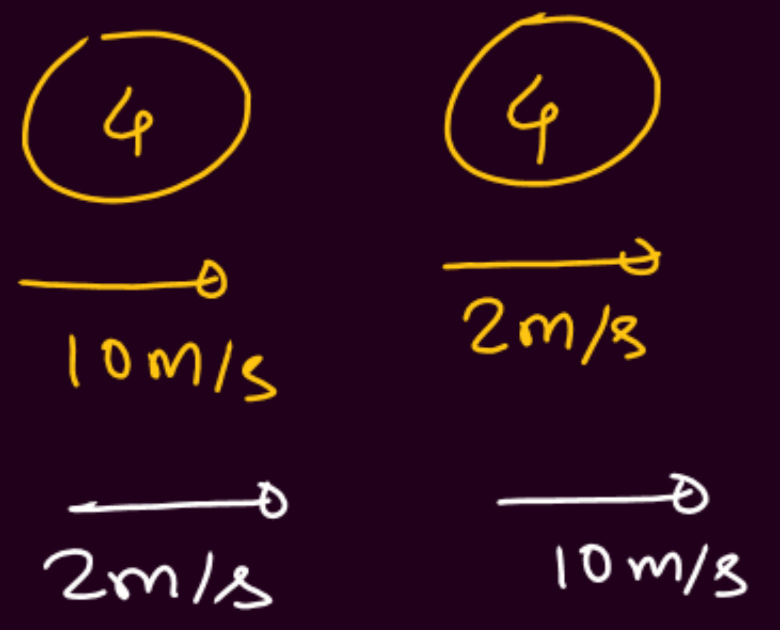
Sp. Case

$(m_1 >> m_2)$

① $v_1 \simeq u_1$

② $v_2 \simeq 2u_1 - u_2$

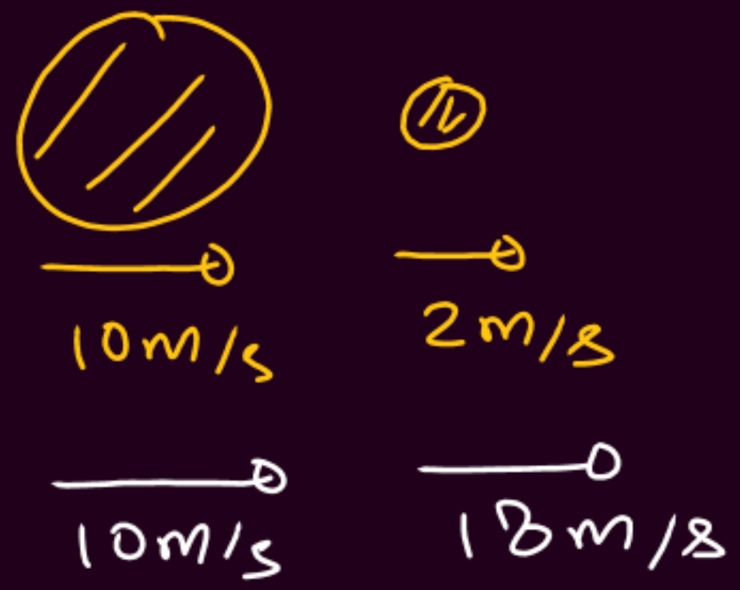
Q.



$e=1$

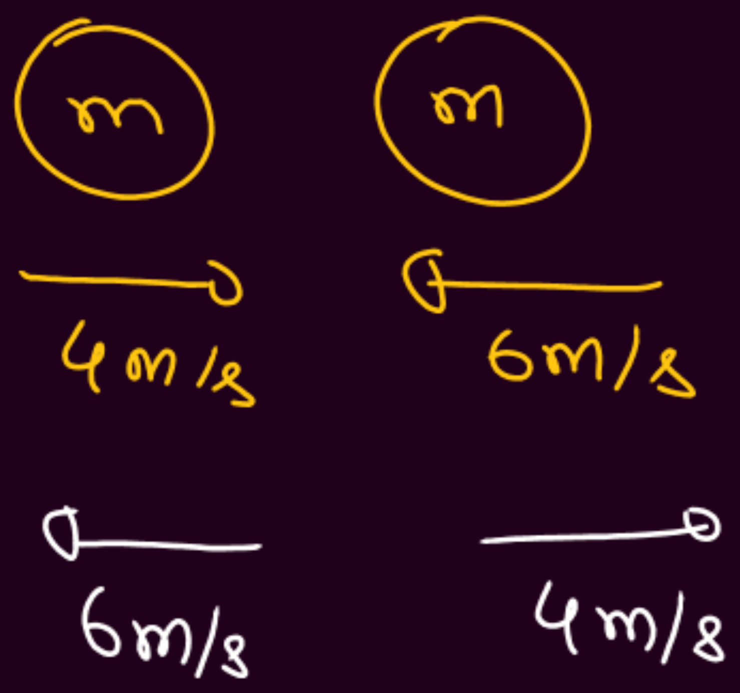
Q.

$e=1$



$v_2 = 2(10) - 2$

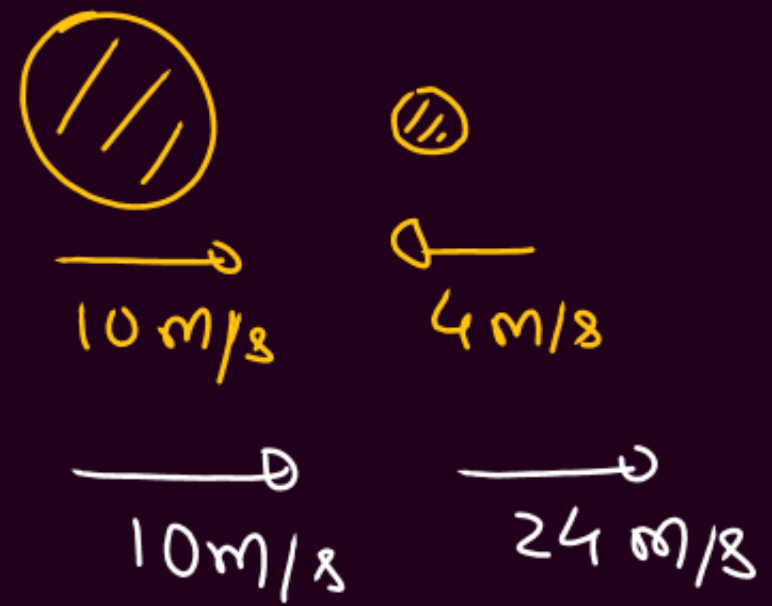
Q



$e=1$

Q.

$e=1$



$v_2 = 2(10) - (-4)$

Q.

$e = 1$

	4	6
Before	$\xrightarrow{\ }$ 10 m/s	$\xrightarrow{\ }$ 5 m/s
After	$\xrightarrow{\ }$ 4 m/s	$\xrightarrow{\ }$ 9 m/s

$$v_1 = 10 + \frac{2(6)}{4+6}(5-10)$$

$$v_1 = 4$$

$$v_2 = 5 + \frac{2(4)}{4+6}(10-5)$$

$$v_2 = 9\ m/s$$

Q.

$e = 1$

	5	10
Before	$\xrightarrow{\ }$ 20 m/s	$\xleftarrow{\ }$ 5 m/s
After	$\xleftarrow{\ }$ 40/3	$\xrightarrow{\ }$ 35/3

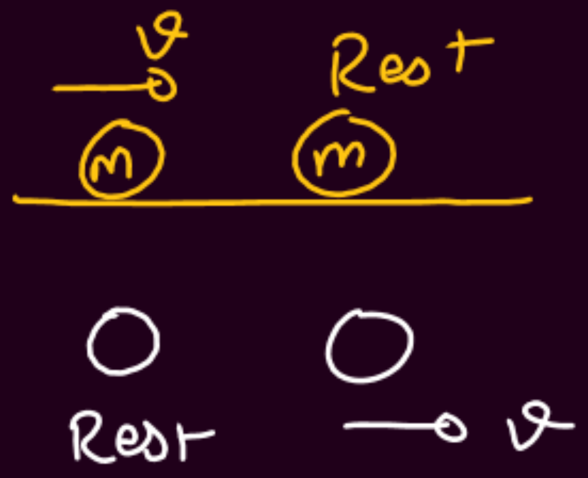
$$v_1 = 20 + \frac{2(10)}{5+10}(-5-20)$$

$$= 20\left[1 - \frac{25}{15}\right] = -\frac{200}{15} = -\frac{40}{3}\ m/s$$

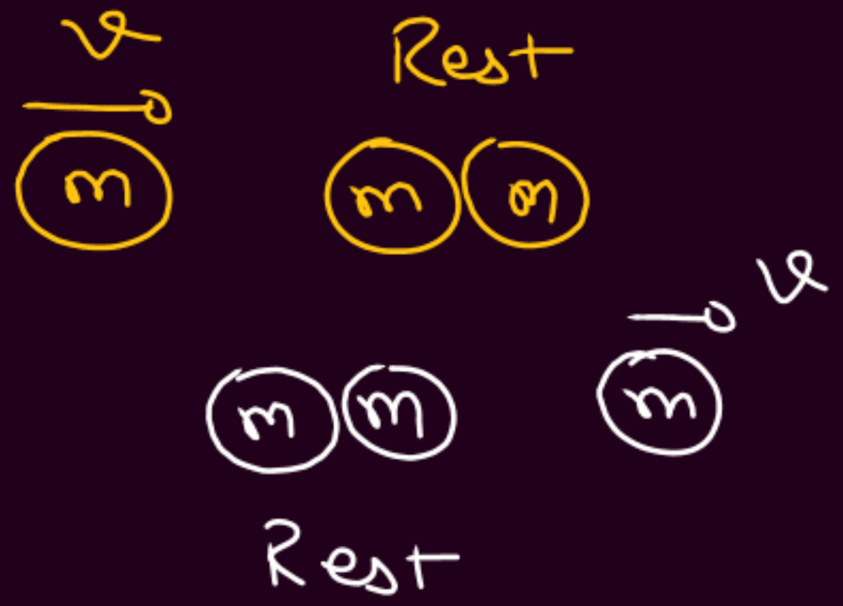
$$v_2 = -5 + \frac{2(5)}{5+10}\left(20-(-5)\right)$$

$$= -5 + \frac{50}{3} = \frac{35}{3}\ m/s$$

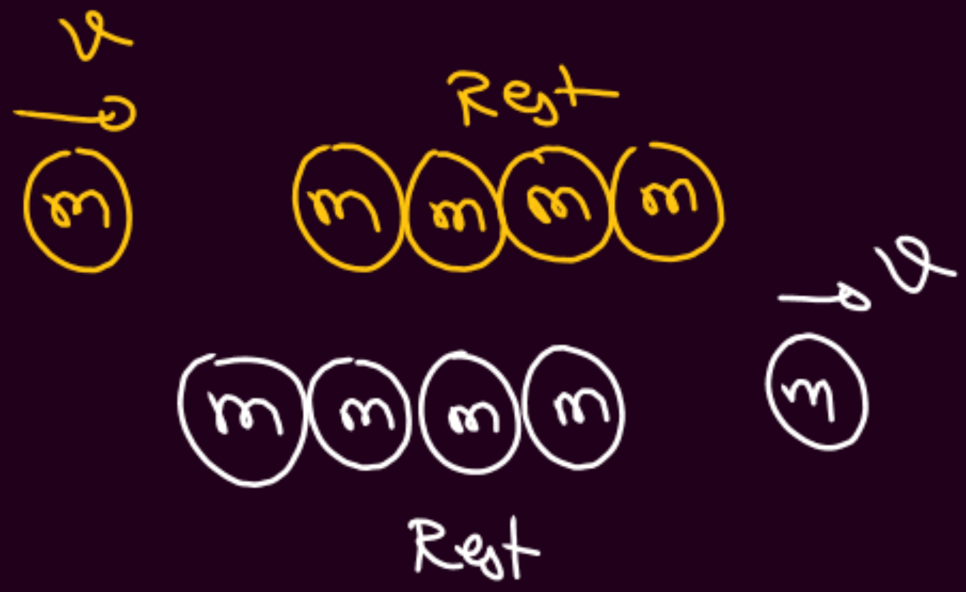
Q


v
Rest
m
m
Rest
v

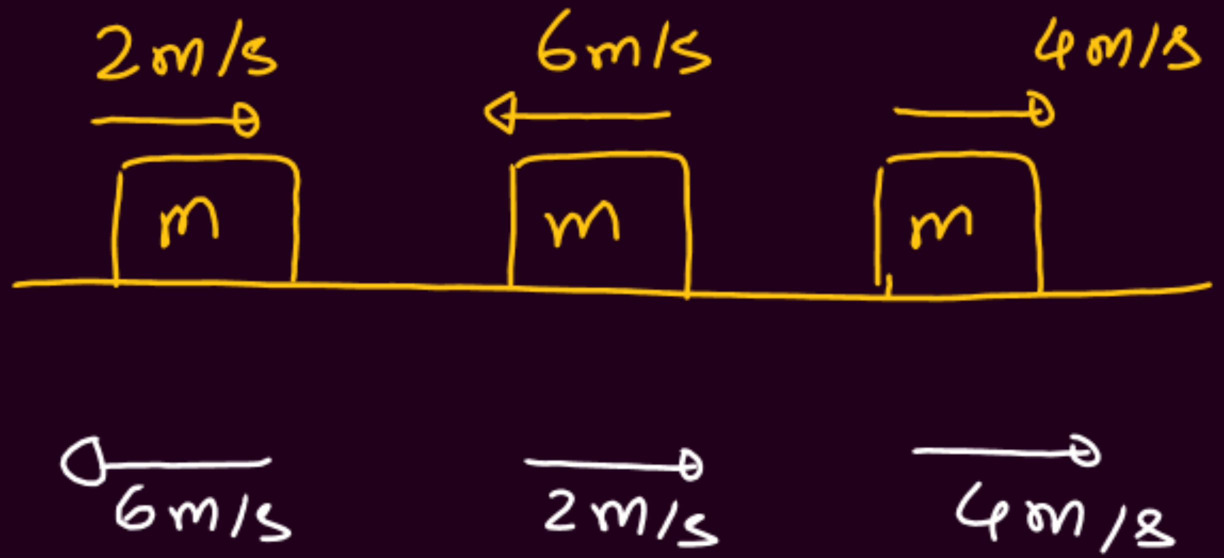
Q.


v
Rest
m
m
m
v
m
m
m
Rest

Q.

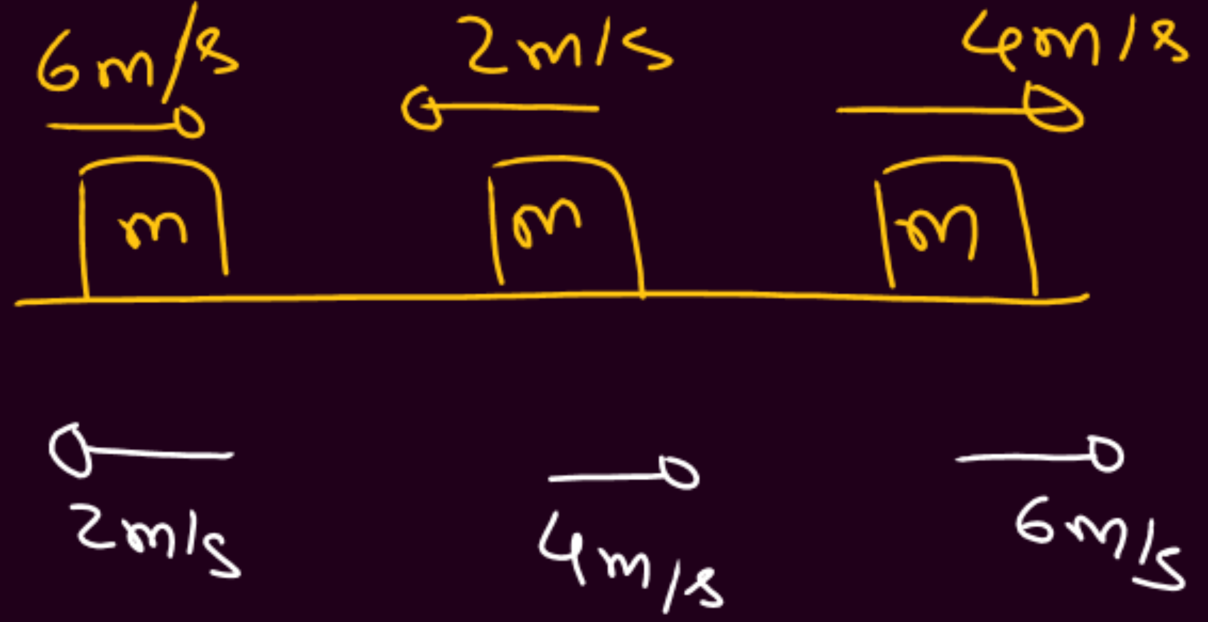

v
Rest
m
m
m
m
m
v
m
m
m
m
m
Rest

Q


2m/s
6m/s
4m/s
m
m
m
6m/s
2m/s
4m/s

$e=1$

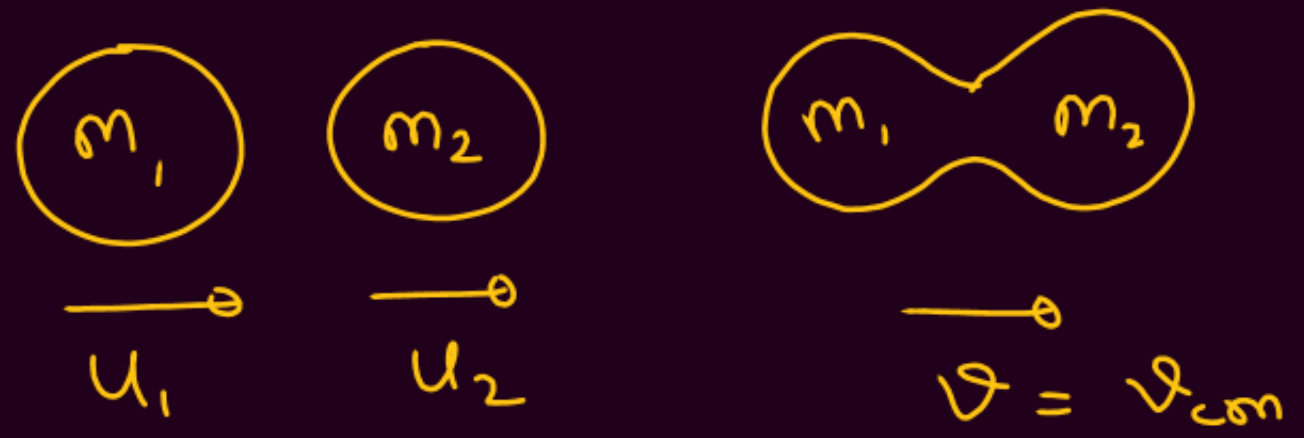
Q.


6m/s
2m/s
4m/s
m
m
m
2m/s
4m/s
6m/s

$e=1$

Head-on Perfectly Inelastic Collision $(e=0)$

→ Stick



① $m_1u_1 + m_2u_2 = (m_1+m_2)v$

$$\boxed{v = \frac{m_1u_1 + m_2u_2}{m_1+m_2}}$$

② $\left(\text{Loss in KE}\right) = \frac{1}{2}\frac{m_1m_2}{m_1+m_2}(u_1-u_2)^2$

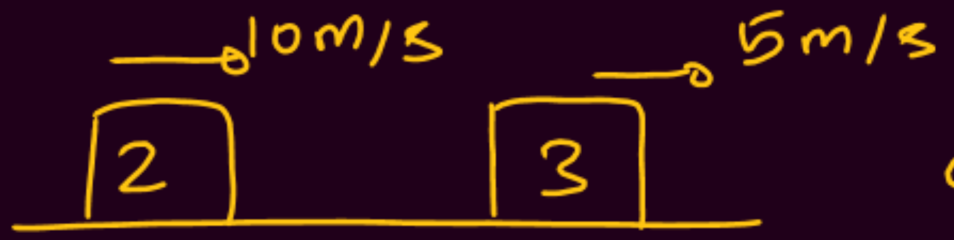
(max)

Sp. case

$$\left(\text{Fractional loss in KE}\right) = \frac{\text{Loss}}{\text{Initial}} = \frac{\frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)(u-0)^2}{\frac{1}{2}m_1u^2}$$

$$= \frac{m_2}{m_1+m_2}$$

Q

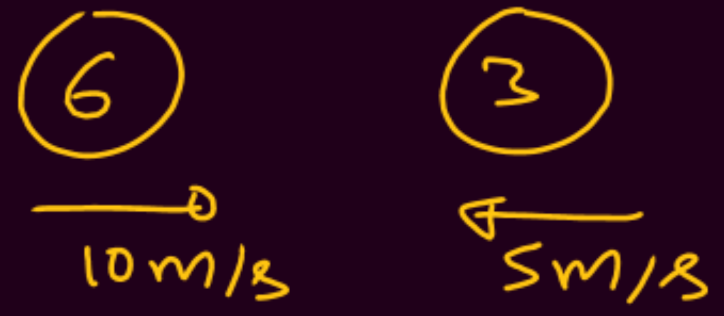


$e = 0$

① $v = \dfrac{2(10) + 3(5)}{2+3} = 7 \text{ m/s}$

② $\text{loss in KE} = \dfrac{1}{2}\,\dfrac{(2)(3)}{5}\,(10-5)^2 = 15\,J$

Q.



$e = 0$

① $v = \dfrac{6(10) + 3(-5)}{6+3} = 5 \text{ m/s}$

② $\left(\text{Loss in KE}\right) = \dfrac{1}{2}\,\dfrac{(6)(3)}{6+3}\left(10-(-5)\right)^2 = \underline{225\,J}$

Q.

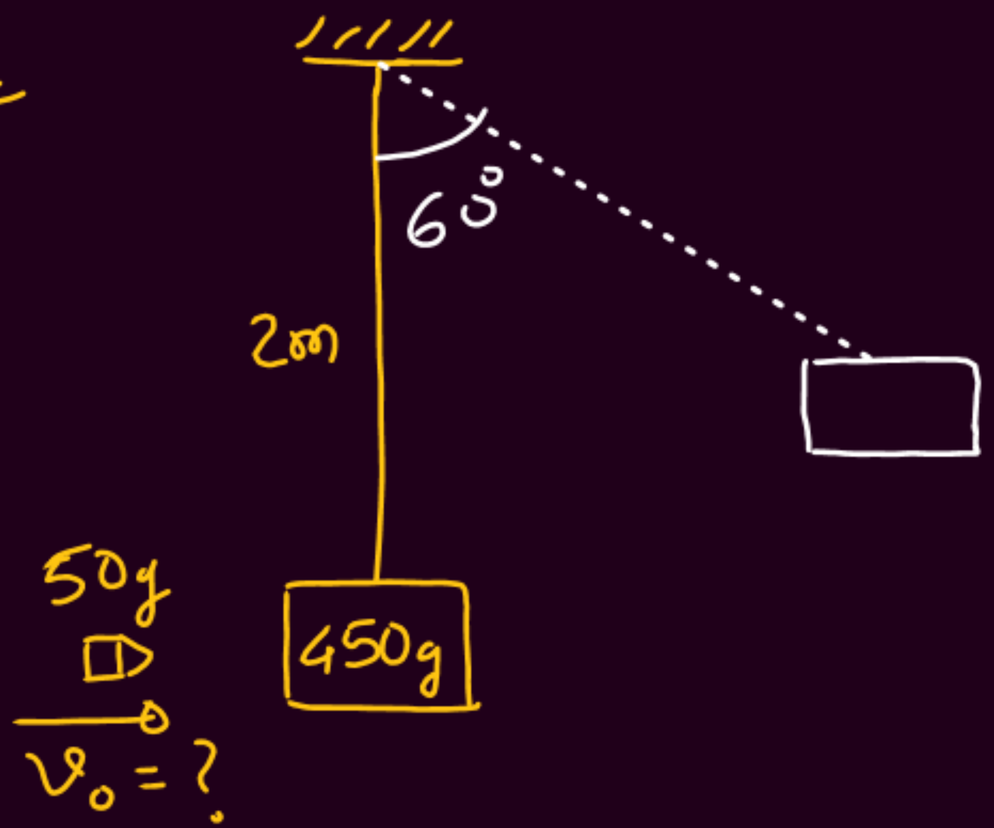
$(e=0)$

80 % loss, $m = ?$

$$\frac{80}{100} = \frac{m}{m+5}$$

$$\boxed{m = 20\text{ kg}}$$

Q.



① $h = R(1-\cos\theta) = 2(1-\cos 60^\circ) = 1\text{ m}$

② $v = \sqrt{2gh} = \sqrt{2(10)(1)} = 2\sqrt{5}\text{ m/s}$

③ $(50)v_0 + 0 = (50+450)\,2\sqrt{5}$

$$v_0 = 20\sqrt{5}\text{ m/s}$$

Head-on Inelastic Collision

① $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$

② $v_2 - v_1 = e(u_1 - u_2)$

③ $$\boxed{v_1 = u_1 + \frac{(1+e)m_2}{m_1+m_2}(u_2-u_1)}$$

$$v_2 = u_2 + \frac{(1+e)m_1}{m_1+m_2}(u_1-u_2)$$

④ $\begin{pmatrix}\text{Loss in}\\ \text{KE}\end{pmatrix} = \frac{1}{2}\,\frac{m_1 m_2}{m_1+m_2}(u_1-u_2)^2(1-e^2)$

Q.

$m = 6$, $m = 4$, $e = \frac{1}{2}$

$10\,m/s \rightarrow$ $\quad 4\,m/s \rightarrow$

$$① \; v_1 = 10 + \frac{\left(1+\frac{1}{2}\right)(4)}{6+4}(4-10)$$

$$= 10 + \frac{(6)(-6)}{10}$$

$$v_1 = 6.4\ m/s$$

$$② \; v_2 = 4 + \frac{\left(1+\frac{1}{2}\right)(6)}{6+4}(10-4)$$

$$= 4 + \frac{9}{10}(6)$$

$$= 9.4\ m/s$$

Sp. Case

m $\quad m$ $\quad e$

$u \rightarrow$ $\quad$ Rest

$\left(\frac{1-e}{2}\right)u \rightarrow$ $\quad \left(\frac{1+e}{2}\right)u \rightarrow$

$$\boxed{\frac{v_1}{v_2} = \frac{1-e}{1+e}}$$

$$v_1 = u + \frac{(1+e)m}{2m}(0-u) = \left(\frac{1-e}{2}\right)u$$

$$v_2 = 0 + \frac{(1+e)m}{2m}(u-0) = \left(\frac{1+e}{2}\right)u$$

Q

4   6

$4\,m/s$ $1\,m/s$

       $3\,m/s$

① $v_1 = ?$

② $e = ?$

COLM

① $4(4) + 6(1) = 4v_1 + 6(3)$

$$v_1 = 1\ m/s$$

② $e = \dfrac{v_2 - v_1}{u_1 - u_2} = \dfrac{3-1}{4-1} = 2/3$

Q

$100g$ $2kg$ $100\,m/s$

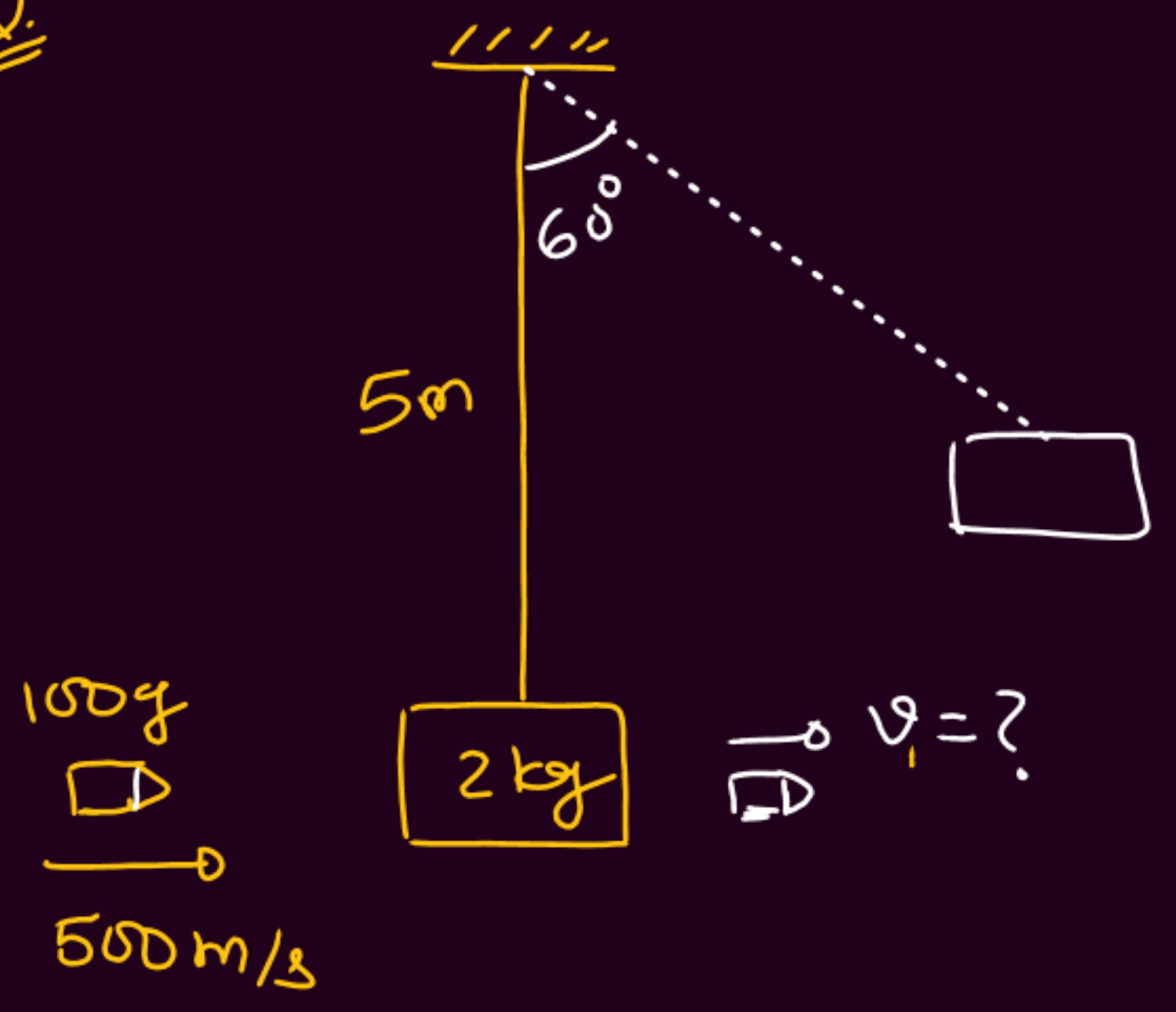
$400\,m/s$ Rest

$$(0.1)\,400 + 2(0) = (0.1)\,100 + 2v$$

$$40 = 10 + 2v$$

$$\boxed{v = 15\ m/s}$$

Q.



① $h = R(1-\cos\theta) = 5(1-\cos 60^\circ) = 2.5\,m$

② $v_2 = \sqrt{2gh} = \sqrt{2(10)(2.5)} = \sqrt{50} = 5\sqrt{2}\ m/s$

③ <u>COLM</u>

$$(0.1)(500) + 0 = (0.1)v_1 + 2(5\sqrt{2})$$

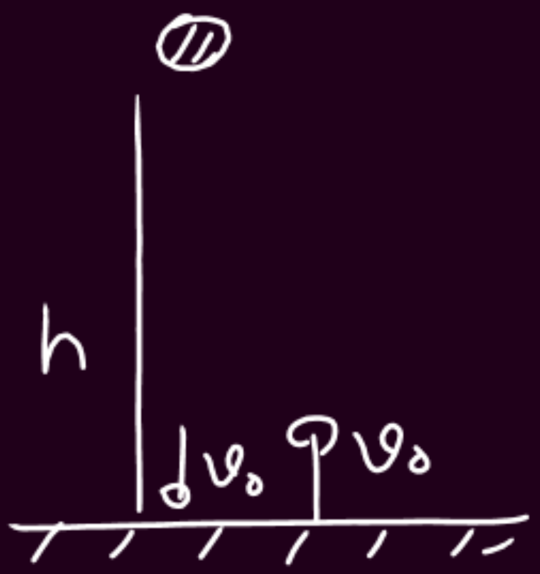
$$50 = (0.1 v_1) + 14$$

$$36 = 0.1 v_1$$

$$\boxed{v_1 = 360\ m/s}$$

Bounceing of Ball

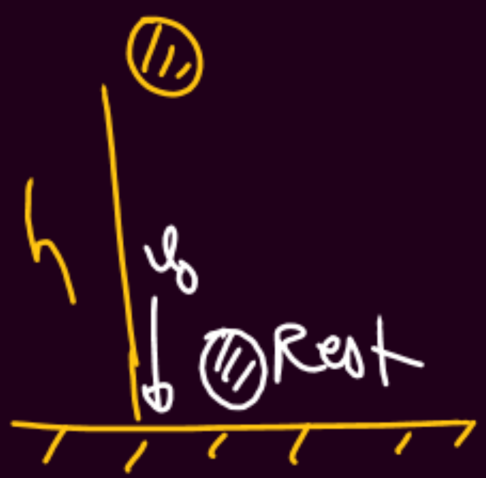
① Elastic Collision



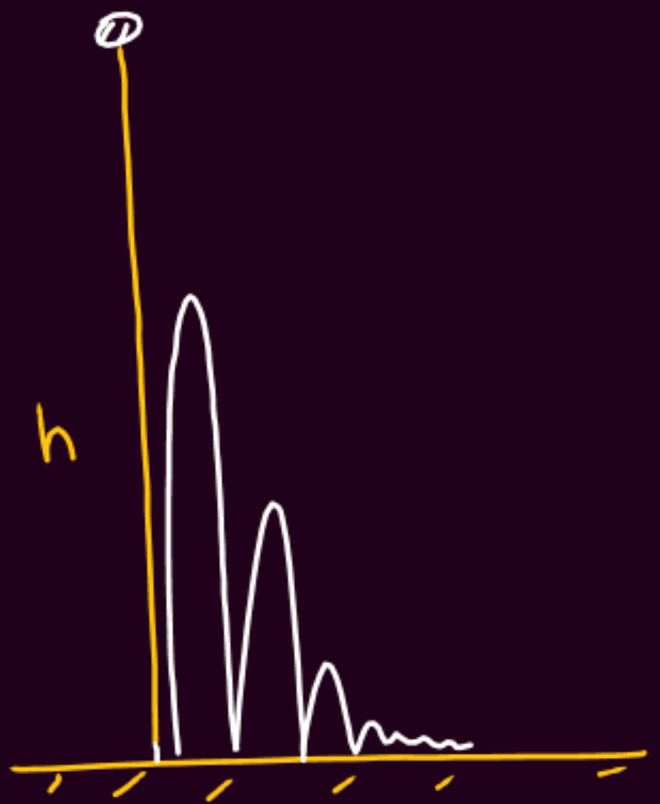
→ Bounces back to same height

→ infinite collision

② Perfectly Inelastic



③ Inelastic



① Vel. just after 1^{st} collision $\boxed{v_1 = e v_0}$

② — " —————— 2^{nd} — " — $v_2 = e^2 v_0$

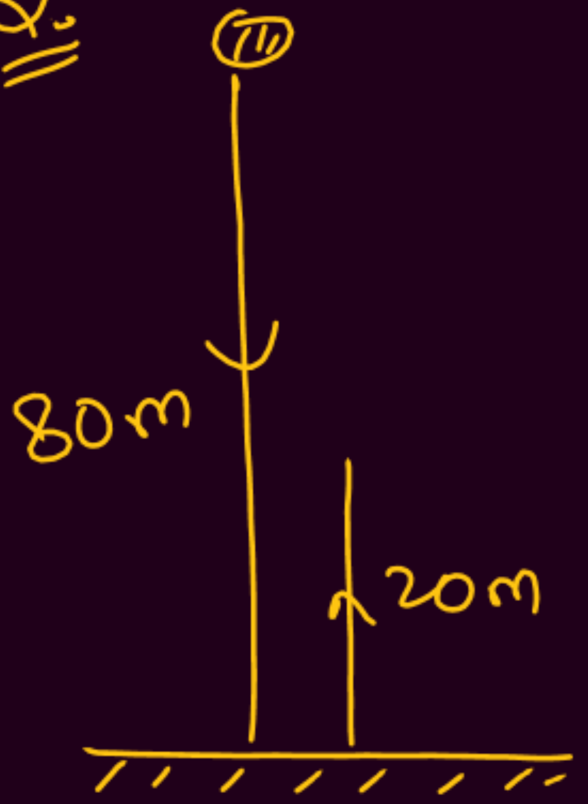
③ — " —————— n^{th} collision $\boxed{v_n = e^n v_0}$

① Max. height after 1^{st} collision $\boxed{h_1 = e^2 h_0}$

② — " —————— 2^{nd} — " — $h_2 = e^4 h_0$

③ —— " —— n^{th} — " — $\boxed{h_n = e^{2n} h_0}$

Q.



$e = ?$

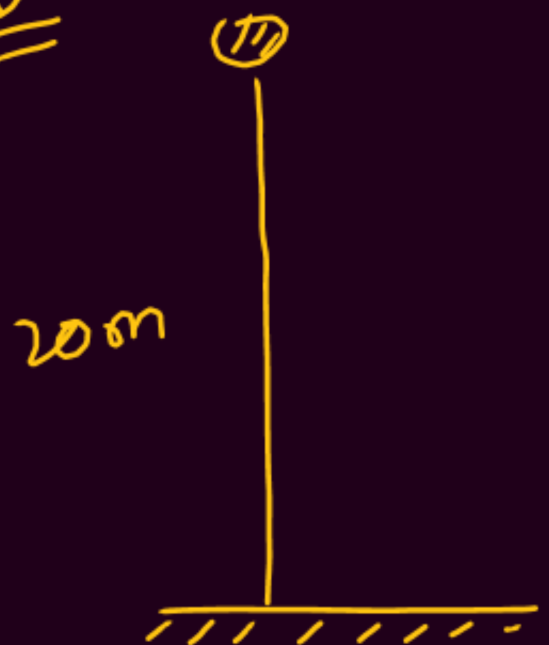
$h_1 = e^2 h_0$

$20 = e^2 (80)$

$e^2 = \frac{1}{4}$

$\boxed{e = \frac{1}{2} = 0.5}$

Q.



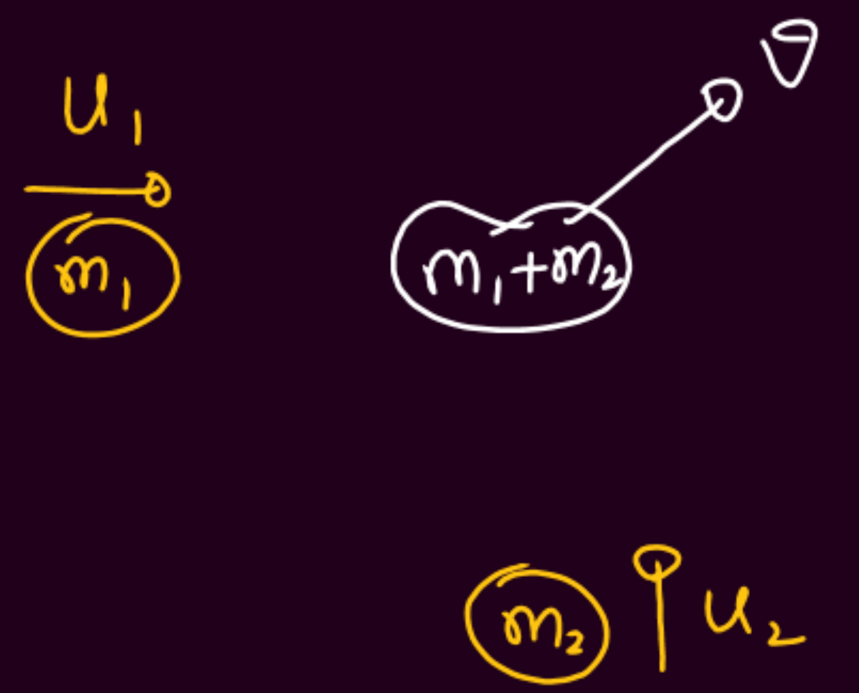
$e = 1/4$

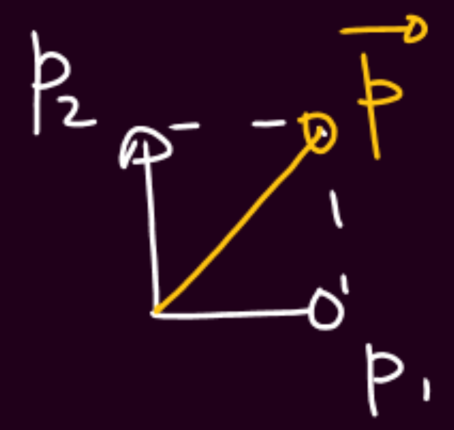
$v_0 = \sqrt{2(10)(20)} = 20\ m/s$

$v_1 = e\, v_0$

$v_1 = \frac{1}{4}(20) = 5\ m$

Case

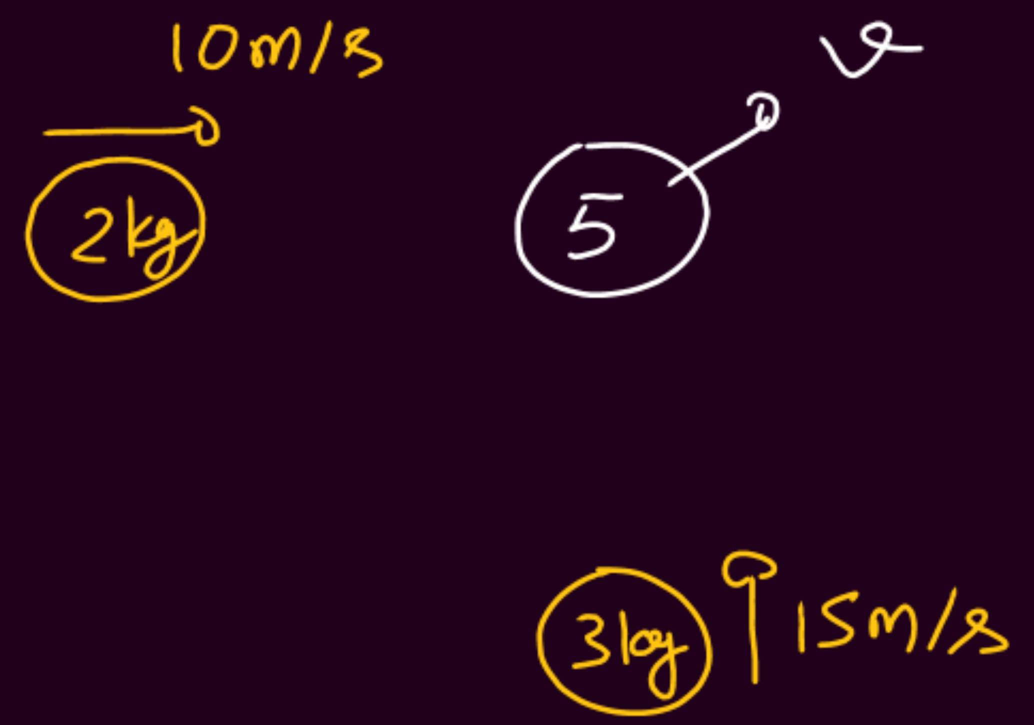




$$\boxed{p^2 = p_1^2 + p_2^2}$$

$$(m_1+m_2)^2 v^2 = (m_1 u_1)^2 + (m_2 u_2)^2$$

Q.



$$(5v)^2 = (10\times 2)^2 + (3\times 15)^2$$

$$v^2 = (2\times 2)^2 + (3\times 3)^2 = 16 + 81 = 97$$

$$\boxed{v = \sqrt{97}\ m/s}$$

Oblique Collision (2D)

COLM $\boxed{\text{Vector}}$

$$m_1 \vec{u_1} + m_2 \vec{u_2} = m_1 \vec{v_1} + m_2 \vec{v_2}$$

If elastic

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Sp. Case

- spherical
- identical
- stationary
- obliquely
- Elastically

① ⊥ dirⁿ

② $\boxed{u^2 = v_1^2 + v_2^2}$

Q.

After collision $v/3$ at θ

$v_2 = ?$

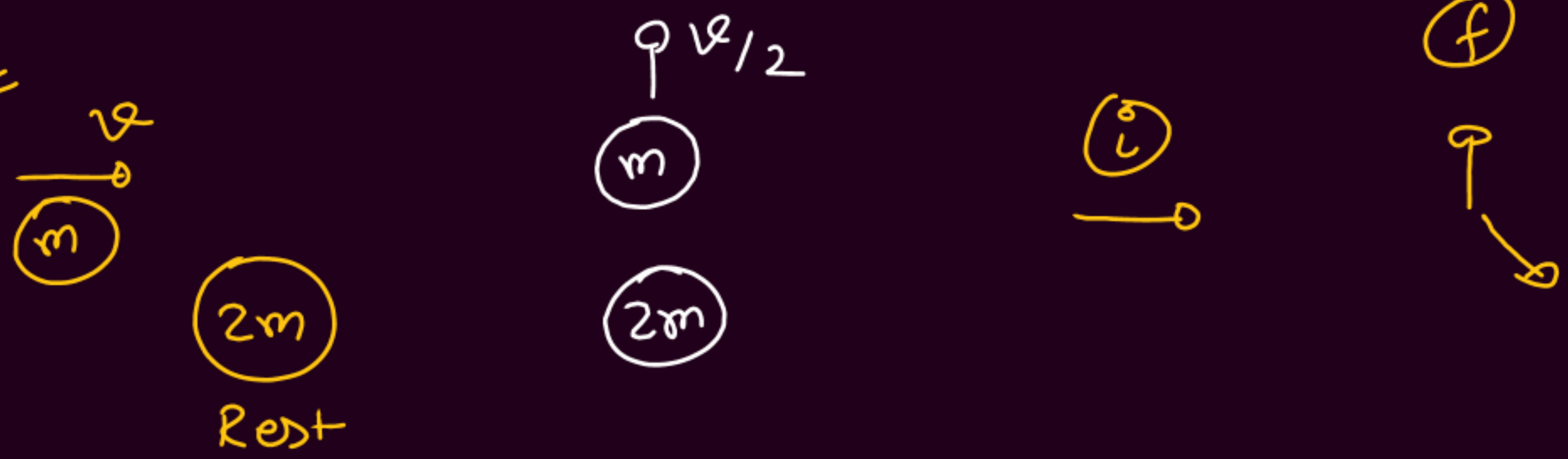
$$u^2 = v_1^2 + v_2^2$$

$$v^2 = \left(\frac{v}{3}\right)^2 + v_2^2$$

$$\frac{8}{9}v^2 = v_2^2$$

$$\boxed{v_2 = \frac{2\sqrt{2}}{3}v}$$

Q.



COLM

$$m(v\hat{i}) + 0 = m\left(\frac{v}{2}\hat{j}\right) + 2m(\vec{v}_2)$$

$$\left(v\hat{i} - \frac{v}{2}\hat{j}\right) = 2\vec{v}_2$$

$$\vec{v}_2 = v\left(\frac{1}{2}\hat{i} - \frac{1}{4}\hat{j}\right)$$

$$v_2 = v\sqrt{\frac{1}{4} + \frac{1}{16}} = \frac{v\sqrt{5}}{4}$$